strain. The usual sex balance has become disturbed. The cause or method of this disturbance is not definitely known. But whatever the cause, the disturbance influenced not only the individual but also its germ plasm and the disturbed balance is evident throughout succeeding generations. The origin of this disturbance of the sexual balance may be referred to as a mutation. While there is little evidence concerning its cause, there seems abundant evidence concerning its permanent character so far as this strain is concerned.

The derivation from this sex intergrade strain of several strains which produce only *normal* (so far as may be judged by their morphological characters at any rate) females, and, on occasion, *normal* males, is a phenomenon similar to that of the sudden appearance of the sex intergrade strain and might with equal propriety be called a return mutation.

I am inclined to believe from evidence from many Cladocera, and from other forms reproducing parthenogenetically during most of the time and by means of sexual reproduction at irregular and uncertain intervals, that environmental factors in all such forms wield the determining influence. The evidence at hand in the present case, however, is not very conclusive and must be reserved for the larger paper.

[1] The following references may be cited:

Goldschmidt, R., Erblichkeitsstudien an Schmetterlingen. I, *Zs. ind. Abs.—Vererbungslehre*, **7**, 1-62 (1912); and a preliminary report on further experiments in inheritance and determination of sex, these PROCEEDINGS, **2**, 53–58.

Riddle, Oscar, Statement run in the *Carnegie Inst. Washington Year Book*, **12**, 322 (1913); and Sex control and known correlations in pigeons, *Amer. Nat.*, **50**, 385–410 (1916).

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION: A REMARKABLE TRIGONOMETRICAL SERIES

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1. The title of this note is perhaps not very appropriate: we retain it because the contents of the note form a natural sequel to those of three papers which we have published under same title elsewhere,[1] and in particular those of our second paper in the *Acta Mathematica*. We there discussed in detail the series

$$\sum e^{\alpha\pi i n^2 + 2\theta\pi i n}, \tag{1.1}$$

and other similar series associated with the elliptic Theta-functions, and used our results to elucidate a variety of difficult points in the theory

of Taylor's series and trigonometrical series. We have since discovered that even simpler and more elegant illustrations may be derived from the series

$$\sum e^{\alpha\pi i n \log n + 2\theta\pi i n}. \qquad (1.2)$$

This series behaves, for different values of the parameters α and θ, far more regularly than does the series (1.1). To put the matter roughly, the behaviour of the series *does not, in its most essential features, depend upon the arithmetic nature of* α.

2. Our fundamental formula is

$$\sum_0^\infty a^{\rho n} e^{-ya^n} = \frac{1}{\log a} \sum_{-\infty}^{\infty} \Gamma\left(\rho + \frac{2\pi i n}{\log a}\right) y^{-\rho - \frac{2\pi i n}{\log a}} - \sum_0^\infty \frac{(-y)^n}{n!\,(a^{\rho+n}-1)}. \qquad (2.1)$$

Here $a > 1$, ρ is real, and $\mathrm{R}\,(y) > 0$. The formula becomes illusory when ρ is zero or a negative integer, but the alterations required are of a trivial character. The formula is easily proved by means of Cauchy's Theorem: similar formulae were proved by one of us in a paper published in 1907.[2]

We now write $y = \sigma + it$, where $t > 0$, suppose that $\sigma \to 0$, and approximate to the series of Gamma-functions by means of Stirling's Theorem. We thus obtain

$$\frac{Ht^{-\rho}}{\log a} e^{-\frac{1}{4}\pi i} f(z) = F(\sigma) + \phi(\sigma), \qquad (2.2)$$

where

$$f(z) = \sum_1^\infty n^{\rho - \frac{1}{2}} e^{\alpha i n \log n} z^n; \qquad (2.21)$$

$$\alpha = \frac{2\pi}{\log a},\ H = \frac{(2\pi)^\rho}{(\log a)^{\rho+\frac{1}{2}}},\ z = re^{i\theta},\ r = e^{-\alpha\sigma/t},\ \theta = \alpha \log\left(\frac{\alpha}{et}\right), \qquad (2.22)$$

so that $r \to 1$ when $\to\sigma\ 0$;

$$F(\sigma) = \sum_0^\infty a^{\rho n} e^{-(\sigma+it)\,a^n}; \qquad (2.23)$$

and $\phi\,(\sigma)$ is of one or other of the forms

$$A + o\,(1),\ O\left(\log \frac{1}{\rho}\right),\ O\,(\sigma^{-\rho+\frac{1}{2}}),$$

according as $\rho < \frac{1}{2}$, $\rho = \frac{1}{2}$, or $\rho > \frac{1}{2}$.

3. It is known[3] that, if $\rho > 0$,

$$F(\sigma) = O\,(\sigma^{-\rho}),\ F(\sigma) = \Omega\,(\sigma^{-\rho}), \qquad (3.1)$$

when $\sigma \to 0$, the second of these formulae meaning[4] that $F(\sigma)$ is *not* of the form $o(\sigma^{-\rho})$, and the two together that

$$0 < h = \overline{\lim}\, \sigma^{\rho} F(\sigma) < \infty. \tag{3.2}$$

These relations all hold uniformly in t. It follows that, if $\rho > 0$ and $r = |z| \to 1$, *the function* $f(z)$ *is exactly of the order* $(1-r)^{-\rho}$, *and this uniformly in* θ. Incidentally, of course, it follows that every point of the unit circle is a singular point: but this is known already.[5]

The series furnishes an example in which the orders in the unit circle of the functions $f(z) = \Sigma\, a_n z^n$ and $g(z) = \Sigma\, |a_n|\, z^n$ differ by exactly $\frac{1}{2}$, the maximum possible.[6]

When $\rho = 0$, $f(z)$ is bounded, but does not tend to a limit when z approaches any point of the unit circle along a radius vector. We know of no other example of a function possessing this property. When $\rho < 0$, $f(z)$ is continuous for $|z| \leq 1$.

4. Let

$$s_n = \sum_{1}^{n} k^{\rho-\frac{1}{2}}\, e^{\alpha i k \log k + 2\theta\pi i k}; \tag{4.1}$$

and suppose first that $\rho > 0$. Then it is easy to deduce from the results of §3 that s_n is of the form $\Omega(n^{\rho})$ when $n \to \infty$. The corresponding 'O' result lies a little deeper: all that can be proved in this manner is[7] that $s_n = O(n^{\rho} \log n)$. But a direct investigation, modelled on that of the early part of our second paper in the *Acta Mathematica*, shows that the factor $\log n$ may be omitted. It should be observed that an essential step in our argument depends on an important lemma due to Landau,[8] according to which

$$\left| \int_{1}^{X} x^{\gamma}\, e^{ix \log(\eta x)}\, dx \right| < 23\, X^{\gamma+\frac{1}{2}} \tag{4.2}$$

for $X \geq 1$, $\gamma \geq 0$, $\eta > 0$. We thus find that s_n *is, for every positive value of* α, *exactly of the order* n^{ρ}, *and this uniformly in* θ. *The series*

$$\sum n^{\rho-\frac{1}{2}}\, e^{\alpha i n \log n + 2\theta\pi i n} \tag{4.3}$$

is never convergent, or summable by any of Cesàro's means.

When $\rho = 0$, s_n is bounded, but the series is never convergent or summable. When $\rho < 0$ it is convergent; and uniformly in θ.

5. For further applications it is necessary to consider the real and imaginary parts of our function and series separately, and this is most easily effected by introducing some restriction as to the value of α. Suppose that a is an integer, not of the form $4k+1$. Thus we may take $a = 2$, $\alpha = 2\pi/\log 2$. Then the results of §§3–4 hold for the real and

imaginary parts of the function or the series. In particular *the series*

$$\sum n^{\rho-\frac{1}{2}} \cos\,(\alpha n \log n + 2\theta\pi n) \quad (\rho \geq 0) \tag{5.1}$$

is never convergent or summable for any value of θ, and is accordingly not a Fourier's series. We thus obtain a solution of what, in our former paper, we call Fatou's[9] problem which combines all the advantages of those given previously by Lusin,[9] Steinhaus,[9] and ourselves.

We can also obtain in this manner exceedingly elegant examples of continuous non-differentiable functions. Thus *the function*

$$f\,(\theta) = \sum \frac{\sin\,(\alpha n \log n + 2\theta\pi n)}{n^{\beta}} \quad (1 < \beta \leq \tfrac{3}{2}) \tag{5.2}$$

does not possess a finite differential coefficient for any value of θ.

[1] G. H. Hardy and J. E. Littlewood, Some problems of Diophantine approximation (i) *Proc. Fifth Int. Congress Math.*, Cambridge, **1**, 223–229 (1912); (ii) *Acta Math.*, **37**, 155–190 (1914); (iii) *Ibid.*, 193–238.

[2] G. H. Hardy, On certain oscillating series, *Quarterly J. Math.*, **38**, 269–288 (1907).

[3] G. H. Hardy, Weierstrass's non-differentiable function, *Trans. Amer. Math. Soc.*, **17**, 301–325, (1916).

[4] *l. c. supra* (1) (iii), p. 225.

[5] G. N. Watson, The singularities of functions defined by Taylor's series, *Quarterly J. Math.*, **42**, 41–53 (1911).

[6] G. H. Hardy: (i) A theorem concerning Taylor's series, *Ibid.*, **44**, 147–160 (1913); (ii) Note in addition to a theorem on Taylor's series, *Ibid.*, **45**, 77–84 (1914).

[7] Cf. E. Landau, Abschätzung der Koeffizientensumme einer Potenzreihe: (i) *Arch. Math. Physik*, ser. 3, **21**, 42–50 (1913); (ii) *Ibid.*, 250–255; (iii) *Ibid.*, **24**, 250–260 (1915).

[8] E. Landau, Uber die Anzahl der Gitterpunkte in gewissen Bereichen, *Göttinger Nachrichten*, 687–771 (p. 707), (1912).

[9] For references see p. 232 of our paper (1) (iii).

STERIC HINDRANCE AND THE EXISTENCE OF ODD MOLECULES (FREE RADICALS)

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The discovery of certain unpredicted facts in organic chemistry has led to the employment of the elusive phrase 'steric hindrance,' a phrase, however, which seems too vague in its significance to connote a real scientific theory. If a steric influence upon a chemical reaction be defined as one which is due to the room occupied by a large atom or group of atoms, such a definition leaves an opportunity for that kind of confusion, which is too frequently found in chemical literature, between